Vector Wave Equation 2-D–FDTD Method for Guided Wave Problems

Michał Okoniewski, *Member, IEEE*

Abstract—**A new compact FDTD algorithm for the full wave analysis of inhomogeneous wave guiding structures, using two dimensional mesh is proposed. The formulation is based on the vector wave equation, and in contrast with previous approaches, allows for the formulation of the algorithm in a real domain only. Moreover, since only transverse electric fields are used, two real, instead of six complex components have to be updated and stored, and since they are both defined at the same mesh nodes, the treatment of dielectric inhomogeneities is simplified. Numerical examples validating the method are presented.**

I. INTRODUCTION

KNOWLEDGE of the properties of waveguides is of importance in the microwave and millimeter waves techniques. From the variety of different, both numerical and analytical methods developed in the past decades, time domain techniques have emerged as particularly attractive, since they are fast, relatively simple to implement, capable of solving almost arbitrary waveguide geometries, and are well suited for implementation on massively parallel computers.

Waveguides are assumed to be homogeneous along the propagation (say z) direction, and support modes with the propagation constants independent of z. That means that the z derivatives of fields can be replaced by $-j\beta$. This idea has recently been exploited in [2]–[5] where both FDTD and TLM *compact* schemes using a two-dimensional mesh were derived. These papers were followed by [6] where the stability and dispersion properties of the compact FDTD method were considered.

In [1], a different approach was presented. A two-dimensional FDTD method was extended to handle structures, which can be described by a two-dimensional vector wave equation. The authors observed that uniform waveguides belong to this class of boundary value problems and used their technique to compute dispersion characteristics of an image guide.

Except for [1], in all of the aforementioned papers, substitution of $\frac{\partial}{\partial z}$ with $-j\beta$ resulted in a formulation of the algorithm in the complex domain. This doubled computer storage requirements, and substantially intensified numerical computations.

In [1], the complex notation was avoided, but at the expense of two-fold increase in the number of equation which had to be solved.

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The author is with the Department of Electrical and Computer Engineering, University of Victoria, Victoria, BC, Canada V8W 3P6
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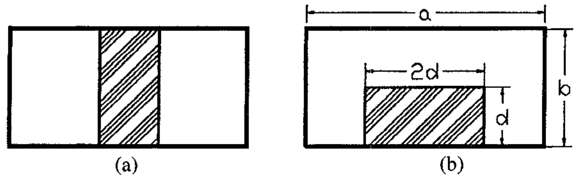

Fig. 1. (a) Dielectric slab loaded rectangular waveguide. (b) Shielded image guide. $\epsilon_r = 2.5$, $k_0 d = 5$, $a/d = 5$, $b/d = 2.5$.

In this letter, a novel compact FDTD method is proposed. It is based on a vector wave rather than Maxwell equations and allows for the formulation of the algorithm in the real domain only. Moreover, it uses only transverse electric field components, hence only two real (but in two consecutive time steps), instead of six complex quantities have to be stored and updated. These features yield substantial computer storage savings.

Another advantage of the method is that, as opposed to the traditional FDTD approaches, both field components used in this formulation are defined at the same mesh nodes, which simplifies the treatment of waveguide inhomogeneities.

II. FORMULATION

Using Maxwell's *curl* equations the following wave equation is obtained:

$$\nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = -\frac{\epsilon_r}{c^2}\frac{\partial^2}{\partial t^2}\vec{E}, \tag{1}$$

where $\epsilon_r = \epsilon_r(x, y)$ is the relative permittivity.

The divergence of the electric field in this formula includes $\frac{\partial}{\partial z}$ term, which can potentially introduce imaginary numbers into the algorithm.

Maxwell's divergence equation can however be used to derive the following relation:

$$\nabla_z E_z = -\frac{1}{\epsilon_r}\nabla_t \cdot \epsilon_r E_t. \tag{2}$$

This relation can now be used in calculations of the divergence term in (1). Thus, first-order z derivatives (responsible for the complex notation in [2]–[6]) can be avoided altogether, and the algorithm be formulated using real numbers only.

Combining (2) and (1), the following vector wave equation is obtained:

$$(\nabla_t^2 - \beta^2)\vec{E} + \nabla(\frac{1}{\epsilon_r}\vec{E}_t \cdot \nabla_t \epsilon_r) = \frac{\epsilon_r}{c^2}\frac{\partial^2}{\partial t^2}\vec{E}, \tag{3}$$

TABLE I
NORMALIZED PHASE CONSTANTS (β/k_0) COMPUTED BY TRM AND ERRORS INTRODUCED BY PROPOSED 2-D–FDTD METHOD RECTANGULAR WAVEGUIDE 22.86 × 10.16mm LOADED WITH A DIELECTRIC SLAB (FIG 1(a)), FOR VARIOUS SLAB WIDTHS AND PERMITTIVITIES MODE TE^x_{10} COMPUTED AT 8 GHz, MODE TE^x_{20} AT 15 GHz

Slab Width		4 mm		8 mm		12 mm		16 mm	
Mode	ϵ	TRM	FDTD error %	TRM	FDTD error %	TRM	FDTD error %	TRM	FDTD error %
TE^x_{10}	2.56	0.9573	–0.25	1.1791	–0.18	1.2957	+0.06	1.3523	+0.19
TE^x_{10}	4.00	1.2485	–0.32	1.5747	+0.02	1.7249	+0.45	1.7956	+0.21
TE^x_{10}	10.00	2.2580	+0.13	2.7617	+0.08	2.9385	+0.36	3.0168	+0.17
TE^x_{20}	2.56	0.5457	–0.13	0.8486	+0.15	1.1267	–0.36	1.2731	+0.30
TE^x_{20}	4.00	0.6177	+0.00	1.2082	+0.34	1.5630	–0.28	1.7218	+0.17
TE^x_{20}	10.00	1.2464	–0.08	2.4978	+0.03	2.8321	+0.12	2.9646	+0.15

from which a pair of coupled equation follows:

$$(\nabla_t^2 - \beta^2)E_x + \nabla_x(\vec{E}_t \cdot \frac{1}{\epsilon_r}\nabla_t\epsilon_r) = \frac{\epsilon_r}{c^2}\frac{\partial^2}{\partial t^2}E_x, \quad (4)$$

$$(\nabla_t^2 - \beta^2)E_y + \nabla_y(E_t \cdot \frac{1}{\epsilon_r}\nabla_t\epsilon_r) = \frac{\epsilon_r}{c^2}\frac{\partial^2}{\partial t^2}\vec{E}_y. \quad (5)$$

Discretization of these equations, using for example central difference scheme, provides the FDTD algorithm. This formulation is particularly useful, when $\epsilon_r(x, y)$ is a continuous function of (x, y) coordinates—$\frac{1}{\epsilon_r}\nabla_t\epsilon_r$ can be precomputed for a given waveguide geometry prior to the actual simulation.

When the waveguide under consideration comprises regions of homogeneous dielectric materials, and thus $\epsilon_r(x, y)$ is a noncontinuous function, (3) may be rewritten so that no gradients of the permittivity are present in the algorithm:

$$(\nabla_t^2 - \beta^2)\vec{E} - \nabla(\nabla_t \cdot \vec{E}_t - \frac{1}{\epsilon_r}\nabla_t \cdot \epsilon_r\vec{E}_t) = \frac{\epsilon_r}{c^2}\frac{\partial^2}{\partial t^2}\vec{E}. \quad (6)$$

It should be noticed that the gradient term in the above equations vanishes inside the homogeneous subregions of the waveguide. This fact, if utilized in numerical implementation, leads to a high speed method (4 multiplications +18 additions per node in a homogeneous region)

Equation (6) yields a pair of coupled equations:

$$(\nabla_x\frac{1}{\epsilon_r}\nabla_x\epsilon_r + \nabla_y^2 - \beta^2)E_x - \nabla_x(\nabla_y - \frac{1}{\epsilon_r}\nabla_y\epsilon_r)E_y = \frac{\epsilon_r}{c^2}\frac{\partial^2}{\partial t^2}E_x, \quad (7)$$

$$(\nabla_y\frac{1}{\epsilon_r}\nabla_y\epsilon_r + \nabla_x^2 - \beta^2)E_y - \nabla_y(\nabla_x - \frac{1}{\epsilon_r}\nabla_x\epsilon_r)E_x = \frac{\epsilon_r}{c^2}\frac{\partial^2}{\partial t^2}E_y. \quad (8)$$

In the numerical implementation (6) is discretized, using central difference scheme. Terms with mixed derivatives are transformed to finite difference with $2\delta x$, $2\delta y$ steps, whereas others using δx,δy. This required ϵ_r to be known in a mesh twice as dense as the one used for E-field, which ensures better simulation of a real structure in the program. The stability condition of this method has not been derived yet, instead, the one introduced in [6] was successfully applied.

TABLE II
PHASE CONSTANTS OF A SHIELDED RECTANGULAR IMAGE GUIDE—FIG 1(b). NUMERICAL RESULTS COMPUTED BY COUPLED MODE METHOD [7], AND RELATIVE DIFFERENCE INTRODUCED BY IEEM–FTT METHOD [8] AND THE PROPOSED 2-D–FDTD METHOD

Mode	CMM [7]	IEEM[8]	FDTD
E^y_{11}	1.529	+0.32%	+0.20%
E^y_{21}	1.459	+0.06%	+ 0.05%

III. NUMERICAL RESULTS

The proposed method was extensively tested, on a number of different structures, ranging from an ordinary rectangular waveguide, to dielectric guides and image guides. There were no convergence problem with air and dielectric loaded rectangular waveguides. A dielectric slab loaded rectangular guide was selected as a good test structure, since its phase constant can be readily computed using transverse resonance method (TRM).

In Table I, phase constants of the slab rectangular waveguides, obtained using TRM and the present method are presented. It is worthwhile to notice that although a relatively coarse mesh was used ($N = 30$) and a high gradient of dielectric properties existed in this structure, the proposed method provided excellent accuracy.

The method was subsequently used to compute phase constants of a dielectric image guide. The results are compared in Table II with the data reported in [7], [8] where coupled mode and IEEM–FFT techniques were used, respectively. Again, though rather coarse mesh was used (30 points in the direction of the wide wall) the result obtained compare well with the published data.

IV. CONCLUSION

In summary, this letter introduces a new compact 2-D–FDTD algorithm. It is based on the vector wave equation, and allows for the formulation of the algorithm in a real domain only. Only transverse electric fields are used, hence two real (in two consecutive time steps) instead of six complex quantities have to be updated and stored. Both transverse field quantities are defined in the same mesh, which simplifies the treatment of the dielectric inhomogeneities. Numerical results validating the method are presented. The method was found numerically stable. A similar formulation can be derived using magnetic rather than electric fields.

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